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Book Review

"Geometric Mechanics. Part II: Rotating, Translating and Rolling", by Darryl D. Holm, Second Edition, Imperial College Press; 2011; ISBN-13: 978-1-84816-778-0 (pbk), USD 31.00

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The second Part of Prof. Holm's book is a masterfully written, very interesting theoretical mechanics text, covering the essence of the relationship between two reductions: Lagrangian symmetry reduction and Hamiltonian reduction. The first leads to the Euler-Poincare′ equation, whereas the second gives a Lie–Poisson equation.

The book comprises twelve chapters and four appendices.

Chapter 1 introduces a principle of Galilean relativity and explains the following notions: a uniformly moving reference frame, the Galilean transformations, the Galilean group, the special Euclidean group (*SE*(*3*)), the matrix representation of *SE*(*3*), Lie algebra of *SE*(*3*), etc. For example, the rigid motion in R^3 refers to a smoothly altering sequence of changes of reference frame along a time-dependent path in *SE*(*3*).

Chapter 2 concisely describes Newtonian, Lagrangian and Hamiltonian approaches to a freely rotating rigid body. A short discussion of a rigid body rotating in a field with a quadratic potential using Manakov's method is also presented.

Chapters 3 and 4 continue the study of freely rotating motion. Because of the close relation of quaternions with rigid-body rotations, the basic philosophy of operating with quaternions is described first. Next, the application of quaternions in Kepler's problem is presented. Then, the quaternionic conjugation is explained (Cayley-Klein parameters, Hopf's fibration, etc). The chapter also introduces the notion of coquaternions in the context of complexified mechanics. Chapter 4, in turn, is intended to provide a comprehensive understanding of the Cayley–Klein dynamics for the rigid body, with a particular emphasis on the Heisenberg Lie group (that is the simplest non-commutative Lie group).

In order to study the rotations and translations in R^3 in more detail, Lie groups and their actions are needed. Chapter 5 defines the adjoint and coadjoint actions of *SO*(3) (*SO*(3) denotes the special orthogonal group). These actions, defined on the appropriate groups studied in chapter 6, provide adequate instruments to derive the Euler-Poincaré equations. In this context, one should read a translation of Poincaré's 1901 fundamental two-page paper, placed in Appendix D.

In chapter 7 a brief derivation of the Euler-Poincaré equations is performed. Then, Kelvin–Noether theorem for this equation for *SE*(3) is formulated and proved. Subsequently, the Kirchoff equations for the motion of an ellipsoidal underwater body, based on vector Euler-Poincaré equations are rederived. Some remarks about the dynamics of the heavy-top prepares the reader for chapter 8.

The heavy-top refers to a rigid body which moves around a fixed point in a gravitational field. Chapter 8 begins with some definitions and the basic equations describing the motion of the top. Thereafter follows the formulation of the heavy-top action principle for a reduced action. The Lie–Poisson description of the heavy-top comes next. At last, the Clebsch heavy-top action principle and the Kaluza-Klein construction are presented and discussed. In both of these approaches, Euler's motion equation for the heavy-top is recovered.

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Chapter 9 focuses on generalizations of action principles on Lie algebras. These include: the Euler-Poincaré theorem, the Hamilton-Pontryagin principle, the Clebsch-Euler-Poincaré approach (which leads to Lie–Poisson Hamiltonian form of the Euler-Poincaré equation), and, briefly, the momentum maps.

Chapter 10 applies the results of the preceding chapter to the systematic presentation of the Lie–Poisson Hamiltonian formulation of the Euler-Poincaré equations for the continuum spin chain, using the specific example of *G*-strands. Let's recall that a *G*-strand denotes a map $g(t, s)$: $R \times R \mapsto G$ for a Lie group G that follows from Hamilton's principle for a certain class of *G*-invariant Lagrangians. The case when $G = SO(3)$, i.e., the $SO(3)$-strand version of the rigid-body equation, may be interpreted physically as a continuous spin chain.

In chapter 11 a brief survey of momentum maps in Poisson geometry, Hamiltonian action and symplectic dual pairs is presented. It summarizes the basic philosophy of the book: *the relation between the results of reduction by Lie symmetry on the Lagrangian and Hamiltonian sides* (p. 242). I think that some fragments are difficult to understand without consulting references (see e.g. Holm D. D., *Applications of Poisson geometry to physical problems*, Geometry and Topology Monographs *17* (2011) 221–384).

Interesting applications of the Hamilton-Pontryagin and Euler-Poincaré approaches to the constrained rocking, rolling, but not sliding motion of a rigid body on a ideally rough horizontal plane are given in chapter 12. Two celebrated, classical nonholonomic problems are considered. The first question concerns the Chaplygin sphere problem, that is, the motion of an inhomogeneous sphere whose center of mass coincides with its geometric center that rolls without slipping on the plane. The second problem deals with the motion of a body that is spinning, rolling and falling, commonly referred to as Euler's disk. The chapter concludes with a discussion of the relations between Lagrange-d'Alembert principle and the Hamilton-Pontryagin and Euler-Poincaré formulations.

Two appendices contain basic material on smooth manifolds, Lie groups and Lie algebras; they are obviously no substitute for advanced books on the mathematical structure of classical mechanics. Yet the appendices help the reader to better understand the notions, notations and so on. Very useful coursework and homework are placed in the third appendix; many exercises are scattered throughout the main text to enable readers to verify their advance.

In sum, I thoroughly recommend this book and believe that it will be a very useful textbook for acquainting students with the fascinating world of geometrical methods in mechanics. The book's content requires from the reader a fairly solid level of mathematical sophistication, but if this requirement is met the reader interested in mechanical problems will be deeply satisfied. Familiarity with *Geometric Mechanics. Part I: Dynamics and Symmetry* is desirable but, in my opinion, not necessary.

The book should surely be valuable not only for advanced students but also for applied mathematicians, specialists in mechanics, geophysicists and theoretical physicists.